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Performance analysis and optimization of straight taper fins with variable heat transfer coefficient

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Abstract

In the present paper, the thermal analysis and optimization of straight taper fins has been addressed. With the help of the Frobenius expanding series the temperature profiles of longitudinal fin, spine and annular fin have been determined analytically through a unified approach. Simplifying assumptions like length of arc idealization and insulated fin tip condition have been relaxed and a linear variation of the convective heat transfer coefficient along the fin surface has been taken into account. The thermal performance of all the three types of fin has been studied over a wide range of thermo-geometric parameters. It has been observed that the variable heat transfer coefficient has a strong influence over the fin efficiency. Finally, a generalized methodology has been pointed out for the optimum design of straight taper fins. A graphical representation of optimal fin parameters as a function of heat duty has also been provided.

1. Introduction

Fins or extended surfaces are widely used to augment the rate of heat transfer from the primary surface to the ambient medium in a large variety of thermal equipment. An accurate analysis of heat transfer in fins has become crucial with the growing demand of high performance of heat transfer surfaces with progressively smaller weights, volumes, initial and running cost of the system. Over the years different fin shapes have been evolved depending upon the application and the geometry of the primary surface. Kern and Kraus [1] have identified three main fin geometries. These are longitudinal fins, radial or circumferential fins and pin fins or spines. For any of the above geometry, fins with straight profile or constant thickness are a common choice as they can be manufactured easily. The thermal design of a constant thickness fin is also relatively simple. However, in any fin the temperature difference reduces from the fin base to fin tip. Accordingly, a saving of fin material can be obtained by progressively narrowing down the fin section. This has initiated a lot of exercises for the determination of optimum fin shapes so that the fin volume is minimum for a given rate of heat dissipation or the rate of heat dissipation is maximum for a given fin volume. The criteria for optimum fin profile under convective conditions was first proposed by Schmidt [2] based on a physical reasoning. Later on Duffin [3] proved Schmidt criteria using calculus of variation. Both Schmidt [2] and Duffin [3] estimated the fin surface area neglecting the profile curvature. This has formed a major assumption in further exercises of fin optimization and is known as length of arc idealization (LAI) in literature. LAI was used for optimizing fin shapes under convecting, radiating, convective-radiating condition [4], for fins with heat generation [5] and for variable thermal conductivity.

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Nomenclature

Bi	Biot number, $h_{av}y_b/k$
h	heat transfer coefficient (W/m^2 K)
h_b	heat transfer coefficient at the fin base (W/m^2 K)
h_{av}	average heat transfer coefficient (W/m^2 K)
h_e	heat transfer coefficient at the tip (W/m^2 K)
J	Jacobian
k	thermal conductivity of fin material (W/m K)
L	length of the fin, see Fig. 1 (m)
m, n	constants used in Eq. (1)
m_p	dimensionless parameter defined in Eq. (4)
q	actual heat transfer rate of a fin (W)
Q	non-dimensional actual heat transfer rate defined in Eq. (20)
q_i	ideal heat transfer rate (W)
Q_i	non-dimensional ideal heat transfer rate defined in Eq. (23)
r_i	base radius, see Fig. 1c (m)
R_i^*	dimensionless radius parameters, $h_{av}r_i/k$
r_o	outer radius of an annular fin (m)
R_o	radius ratio, r_o/r_i
T	local fin temperature (K)
T_a	temperature of the surrounding gas medium of the fins (K)
T_t	tip temperature (K)
T_w	fin base temperature (K)
U	dimensionless fin volume defined in Eq. (25)
V	fin volume (m^3)
x, y	Cartesian coordinates (m)
X, Y	$x/L, y/y_b$ respectively
y_b	fin base semi-thickness (m)
y_e	fin tip semi-thickness (m)
Z_0	fin parameter, $\sqrt{Bi}/\psi$

Greek symbols

α	parameter defined in Eq. (4)
β	tip loss parameter, $2\psi/(1+\varepsilon)$
δ	dimensionless parameter, Bi/ψ
ε	ratio of base to tip heat transfer coefficient, h_b/h_e
η	fin efficiency
λ	ratio of tip to base fin thickness, y_e/y_b
ν	parameter defined in Eq. (4)
ψ	aspect ratio, y_b/L
θ	dimensionless temperature, $(T-T_a)/(T_w-T_a)$
θ_0	excess base temperature, T_w-T_a (K)
θ_t	dimensionless tip temperature, $(T_t-T_a)/(T_w-T_a)$
ξ	dimensionless parameter defined in Eq. (4)

Maday [6] in his pioneering analysis proposed the correct formulation for the optimization of longitudinal fin with the elimination of LAI and obtained a profile much different from Duffin [3]. Guceri and Maday [7] further extended this analysis for radial fins.

However, fin shapes determined by the above procedure are complex and difficult to manufacture. These fins have structurally weak slender tips, which do not substantially contribute to the overall heat dissipation. This has resulted in a parallel effort to design optimum fins where the fin shape is specified a priori and fin dimensions are determined to give maximum heat dissipation for a given fin volume.

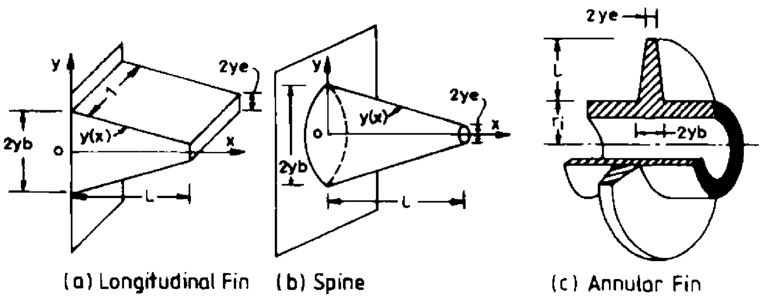

Fig. 1. Geometrical configurations of typical straight taper fins.

Aziz [8] in his review paper thoroughly discussed the state of the art of fin optimization when fin shape is specified. Based on the assumptions proposed by Murray [9] and Gardner [10], he has presented the optimum dimensions of longitudinal fins (straight, triangular and concave parabolic profiles), spines (cylindrical, conical, concave and convex parabolic profiles) and radial fins (straight, trapezoidal and triangular profiles). Finally he discussed the effect of tip heat loss, variable heat transfer coefficient, temperature dependent thermal properties and internal heat generation on the optimal dimensions of a few specific fins.

Chung and Kan [11] considered the effect of profile curvature on the optimum dimensions of longitudinal fins of triangular, concave and convex parabolic profile. While they have proposed an analytical solution for triangular fin they had to take the resort of numerical techniques for parabolic fins. Razelos and Satyaprakash [12] presented an analysis for optimum longitudinal fin of trapezoidal section based on an assumption of negligible heat loss from the fin tip and negligible surface curvature effect and finally suggested a correlation for the optimality criteria. Based on a diameter dependent convective heat transfer coefficient, Chung [13] improved the design of optimum cylindrical pin fins originally proposed by Sonn and Bar-Cohen [14]. Chung and Kan [11] determined the optimum dimensions of spines having different profiles (cylindrical, conical, concave and convex parabolic) from a generalized formulation using a numerical procedure. They reported a profound influence of profile curvature on the optimum dimensions of the spine. Razelos [15] analyzed the heat transfer from convective spines of different profiles assuming negligible surface curvature and no tip loss. Using the Lagrangian multiplier technique the author derived the thermo-geometric criteria for the optimum spines. On the other hand Ulman and Kalman [16] solved the conduction equation for radial fins of different profiles (straight, hyperbolic, triangular and parabolic) numerically to find out the rate of heat dissipation. They obtained the optimum fin dimensions for each of the profiles for the maximum value of heat dissipation under a volume constraint.

Most of the analytical works on fin design and optimization carried out till date are based on the assumption of constant convective heat transfer coefficient along the fin length. However, existence of a non-uniform heat transfer coefficient has been established theoretically and observed experimentally. Kraus [17] has thoroughly discussed the results of the investigations, which have considered non-uniform heat transfer coefficient along the fin surface. He has concluded that non-uniformities have an impact on the rate of heat dissipation by the fins. By a unique experiment Ghai [18] demonstrated that heat transfer coefficient increases towards the fin tip with a minimal value at the fin base. Gardner [19] showed that the variation of heat transfer coefficient along the fin length could be expressed in the form of an equation using Ghai's experimental results. Kraus et al. [20] discussed the findings of Ghai and Gardner in some details. This has triggered a number of investigations considering linear [21], power law [22] and exponential variation [23] of heat transfer coefficient. However, the actual nature of variation of the heat transfer coefficient can be obtained by a conjugate analysis of conduction in the fin along with the convection in the adjacent fluid. Such studies have been taken up by a number of researchers. Stachiewicz [24] reported a general increase of heat transfer coefficient from fin base to fin tip with a marked dip at about 75% of the fin length. Sparrow and Acharya [25] observed a decrease in the heat transfer coefficient near the fin base and a subsequent increase in the down stream for fins under natural convection for a wide range of conditions. Simultaneous, solution of the problem for convection in the fluid and conduction in the fin has also been tried by Advani and Sukhatme [26] and Garg and Velusamy [27].

In a parallel effort, variation of local heat transfer coefficient as a function of local temperature excess has been considered by Unal [28] and Yeh [29]. Such studies have been taken up for their particular relevance to nucleate boiling.

The exact nature of variation of the heat transfer coefficient is not yet established. Nevertheless, efforts have been made for finding out the optimum fin dimensions assuming typical variations of heat transfer coefficients a priori. Razelos and Imre [30] considered the effect of variable heat transfer coefficient and variable thermal conductivity on the optimum dimensions of radial fins with trapezoidal cross section. The authors solved the equation optimality numerically for a length dependent heat transfer coefficient and temperature dependent thermal conductivity. Netrakanti and Huang [31] employed a method of invariant embedding to solve the identical problem. Razelos [15] used the Pontryagin's minimum criteria for finding out minimum mass convective fins. Using variational principle Natarajan and Shenoy [32] determined the optimum profile of conical spines. They considered a power law type dependence of heat transfer coefficient with diameter for different convective conditions. Using hypergeometric function, Yeh [29,33–35] optimized straight fins, spine and fin assemblies for temperature dependent local heat transfer coefficients.

In the present paper, a method has been suggested for optimizing longitudinal, radial and pin fin with straight taper (trapezoidal profile) based on a generalized approach. Simplifying assumptions like LAI and insulated fin tip have been eliminated. A variable convective heat transfer coefficient at the fin surface has been considered. Finally, the fin performance has been obtained in an analytical form so that classical techniques can be adopted for optimization.

2. Mathematical analysis

Fins of three basic geometries, namely, longitudinal, spine and annular with a straight taper are schematically shown in Fig. 1. It is assumed that fins are of constant thermal conductivity and they exchange heat with the ambient medium solely by convection. The generalized differential equation at steady state for above three types of fins can be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y^m (r_\mathrm{i}+x)^n \frac{\mathrm{d}\theta}{\mathrm{d}x}\right] = \frac{m y^{m-1}(r_\mathrm{i}+x)^n h}{k}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\theta \tag{1}$$

In the above equation, $m = 1$, $n = 0$ for longitudinal fin; $m = 2$, $n = 0$ for spine and $m = 1$, $n = 1$ for annular fin.

For straight taper fins the fin profile can be expressed in non-dimensional form as

$$Y = \frac{y}{y_\mathrm{b}} = 1 - (1-\lambda)X \tag{2}$$

where, $\lambda = y_\mathrm{e}/y_\mathrm{b}$ and $X = x/L$.

For a linear variation of the coefficient of convective heat transfer along the fin length and substituting Eq. (2) the governing differential equation can finally be expressed as follows:

$$Y(1-\alpha Y)\frac{\mathrm{d}^2\theta}{\mathrm{d}Y^2} + \{m - \alpha(m+n)Y\}\frac{\mathrm{d}\theta}{\mathrm{d}Y} - m m_\mathrm{p}(1-\alpha Y)(\xi - \nu Y)\theta = 0 \tag{3}$$

where

$$\alpha = \frac{(R_\mathrm{o}-1)}{n(1-\lambda)+(R_\mathrm{o}-1)} = \frac{Bi/\Psi}{nR_\mathrm{i}^*(1-\lambda)+Bi/\Psi},\quad R_\mathrm{i}^* = \frac{h_\mathrm{av}r_\mathrm{i}}{k},\quad h = \frac{2h_\mathrm{av}(\xi-\nu Y)}{(1+\varepsilon)},\quad \xi = \frac{1-\lambda\varepsilon}{1-\lambda},\quad \nu = \frac{1-\varepsilon}{1-\lambda},$$
$$m_\mathrm{p} = \frac{Z_0^2}{(1-\lambda)^2}\sqrt{\left(\frac{2}{1+\varepsilon}\right)^2 + \beta^2(1-\lambda)^2},\quad Z_0 = \frac{1}{\Psi}\sqrt{Bi},\quad \beta = \frac{2\Psi}{1+\varepsilon},\quad \varepsilon = \frac{h_\mathrm{b}}{h_\mathrm{e}},\quad Bi = \frac{h_\mathrm{av}y_\mathrm{b}}{k},\quad \Psi = \frac{y_\mathrm{b}}{L}\quad \text{and}$$
$$R_\mathrm{o} = \frac{r_\mathrm{o}}{r_\mathrm{i}} \tag{4}$$

The above equation is subjected to the following boundary conditions:

$$\text{at } Y = 1,\quad \theta = 1 \tag{5}$$

$$\text{at } Y = \lambda,\quad \frac{\mathrm{d}\theta}{\mathrm{d}Y} = \frac{Z_0^2\beta\theta}{(1-\lambda)} \tag{6}$$

where

$$\beta = \begin{cases} \dfrac{2\Psi}{1+\varepsilon} & \text{for convective heat loss from the fin tip} \\ 0 & \text{when fin tip is insulated(as } \varepsilon\to\infty) \end{cases}$$

With a proper arrangement, Eq. (3) can be reduced in a form such that the method of Frobenius [36] can be applicable. The solution can be obtained in the form of a power series provided the convergence criterion is satisfied. It can be shown that the convergence criteria can be satisfied separately for three specific cases of longitudinal fin, spine and annular fin. Finally the temperature distribution in any of these fins can be expressed as a convergent series. The general expression of temperature distribution for the three types of fins is as follows:

$$\theta(Y) = \left\langle (2-m)(E_3 \ln Y - E_4) + (m-1)E_3 Y^{-1} + \sum_{j=1}^{\infty}[E_3 \ln Y - E_4]A_j Y^{j-m+1} + E_3 \sum_{j=1}^{\infty} C_j Y^{j-m+1} \right\rangle \Big/ (E_2E_3 - E_1E_4) \tag{7}$$

where

$$E_1 = (2-m) + \sum_{j=1}^{\infty} A_j \tag{8}$$

$$E_2 = (m-1) + \sum_{j=1}^{\infty} C_j \tag{9}$$

$$E_3 = \sum_{j=1}^{\infty}(j-m+1)A_j\lambda^{j-m} - \frac{(Z_0^2\beta)}{1-\lambda}\left[2-m+\sum_{j=1}^{\infty}A_j\lambda^{j-m+1}\right] \tag{10}$$

$$E_4 = (3-2m)\lambda^{-m} + \sum_{j=1}^{\infty}A_j\lambda^{j-m} + \ln\lambda\sum_{j=1}^{\infty}(j-m+1)A_j\lambda^{j-m} + \sum_{j=1}^{\infty}(j-m+1)C_j\lambda^{j-m} - \frac{(Z_0^2\beta)}{1-\lambda} \times \left[\ln\lambda\left\{2-m+\sum_{j=1}^{\infty}A_j\lambda^{j-m+1}\right\} + \sum_{j=1}^{\infty}C_j\lambda^{j-m+1} + (m-1)\lambda^{1-m}\right] \tag{11}$$

$$A_0 = 2-m \tag{12}$$

$$A_1 = mm_\mathrm{p}\xi \tag{13}$$

$$A_2 = \frac{(2n\alpha + mm_\mathrm{p}\xi)A_1 - m_\mathrm{p}(n\alpha\xi + mv)A_0}{2(3-m)} \tag{14}$$

$$A_j = \frac{\lfloor nj(j-1)\alpha + mm_\mathrm{p}\xi\rfloor A_{j-1} - m_\mathrm{p}\lfloor(n\alpha\xi + mv)A_{j-2} - n\alpha vA_{j-3}\rfloor}{j(j-m+1)} \quad \text{for } j \geqslant 3 \tag{15}$$

$$C_0 = m-1 \tag{16}$$

$$C_1 = n\alpha - 2m_\mathrm{p}\xi \tag{17}$$

$$C_2 = \langle 2(3-m)\{3n\alpha A_1 + (2n\alpha + mm_\mathrm{p}\xi)C_1 - m_\mathrm{p}(n\alpha\xi + v)C_0\} - (5-m)\{(2n\alpha + m_\mathrm{p}m\xi)A_1 - m_\mathrm{p}(n\alpha\xi + mv)A_0\}\rangle/[2(3-m)]^2 \tag{18}$$

and

$$C_j = \langle j(j-m+1)\{n\alpha(2j-1)A_{j-1} + [n\alpha j(j-1) + mm_\mathrm{p}\xi]C_{j-1} - m_\mathrm{p}(n\alpha\xi + mv)C_{j-2} + nm_\mathrm{p}\alpha vC_{j-3}\} - (2j-m+1) \times \{\lfloor n\alpha j(j-1) + mm_\mathrm{p}\xi\rfloor A_{j-1} - m_\mathrm{p}(n\alpha\xi + mv)A_{j-2} + n\alpha m_\mathrm{p}vA_{j-3}\}\rangle/[j(j-m+1)]^2 \quad \text{for } j \geqslant 3 \tag{19}$$

The rate of heat dissipation from the fin is expressed in an appropriate dimensionless form in the following equation:

$$Q = \frac{h_\mathrm{av}^{m-1}m^{m+n}q}{2^{m+n}r_\mathrm{i}^n\pi^{m+n-1}k^m\theta_0} = \delta^{m-1}\Psi^m(1-\lambda)\left.\frac{\mathrm{d}\theta}{\mathrm{d}Y}\right|_{Y=1} \tag{20}$$

where the temperature gradient at the fin base is given by:

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}Y}\right|_{Y=1} = \Delta = \left\langle E_3\left\{(3-2m) + \sum_{j=1}^{\infty}A_j + \sum_{j=1}^{\infty}(j-m+1)C_j\right\} - E_4\sum_{j=1}^{\infty}(j-m+1)A_j\right\rangle \Big/ [E_2E_3 - E_1E_4] \tag{21}$$

and

$$\delta = \frac{Bi}{\Psi} = Z_0^2 \Psi \tag{22}$$

The rate of heat dissipation from a fin of given geometry will be maximum if the entire fin surface is maintained at its base temperature. In a non-dimensional form the maximum rate of heat dissipation is given by

$$\begin{aligned} Q_{\rm i} &= \frac{h_{\rm av}^{m-1} m^{m+n} q_{\rm i}}{2^{m+n} r_{\rm i}^n \pi^{m+n-1} k^m \theta_0} \\ &= \frac{\delta^m}{\Psi^{1-m}} \left[m \left(\frac{R_{\rm o}-1}{1-\lambda} \right)^n \sqrt{\left(\frac{2}{1+\varepsilon} \right)^2 + (1-\lambda)^2 \beta^2} \left\{ \left(\frac{1}{\alpha} + 1 - m \right) \xi - \left\langle \left(\frac{1}{\alpha} + 1 - m \right) v + (1-m+n) \xi \right\rangle \frac{(1+\lambda)}{2} \right. \right. \\ &\qquad \left. \left. + \frac{v(1-m+n)(1+\lambda+\lambda^2)}{3} \right\} + R_{\rm o}^n \lambda^m \beta \right] \end{aligned} \tag{23}$$

It may be noted that for a longitudinal fin both Q and $Q_{\rm i}$ have been determined considering a unit width of the fin as shown in Fig. 1. Conventionally, the fin efficiency is defined as the ratio of actual heat transfer to the maximum possible heat transfer.

$$\eta = \frac{Q}{Q_{\rm i}} \tag{24}$$

3. Optimum dimensions of straight taper fins

An effort has been made to optimize the dimensions of the three types of straight taper fins based on the above analysis. The fin volume can be expressed in a non-dimensional form.

$$U = \frac{(2m+2n-1)h_{\rm av}^{m+n+1} v}{(n+1)\pi^{m+n-1} k^{m+n+1}} = \frac{\delta^{m+n+1}\Psi^m}{(1-\lambda)^n} \left[\frac{(2-m+2n)(1+\lambda)}{\alpha} + (m-2n-1)(1+\lambda+\lambda^2) \right] \tag{25}$$

In the above expression, the volume corresponds to the unit width of a longitudinal fin. Both the rate of heat transfer and the fin volume are function of δ $(= Bi/\psi)$ and ψ, if other geometrical and thermal parameters are specified. Using Eqs. (20) and (25) the general condition for optimality may be expressed as follows:

$$J = J\left(\frac{Q, U}{\delta, \Psi} \right) = \begin{vmatrix} \frac{\partial Q}{\partial \delta} & \frac{\partial U}{\partial \delta} \\ \frac{\partial Q}{\partial \Psi} & \frac{\partial U}{\partial \Psi} \end{vmatrix} = 0 \tag{26}$$

where

$$\frac{\partial Q}{\partial \delta} = \frac{(m-1)Q}{\delta} + \delta^{m-1}\Psi^m(1-\lambda)\frac{\partial \Delta}{\partial \delta} \tag{27}$$

$$\frac{\partial Q}{\partial \Psi} = \frac{mQ}{\Psi} + \delta^{m-1}\Psi^m(1-\lambda)\frac{\partial \Delta}{\partial \Psi} \tag{28}$$

$$\frac{\partial U}{\partial \delta} = \frac{(m+n+1)U}{\delta} - \frac{(2-m+2n)\delta^{m+n+1}\Psi^m(1+\lambda)}{(1-\lambda)^n \alpha^2} \frac{\partial \alpha}{\partial \delta} \tag{29}$$

$$\frac{\partial U}{\partial \Psi} = \frac{(mU)}{\Psi} - \frac{(2-m+2n)\delta^{m+n+1}\Psi^m(1+\lambda)}{(1-\lambda)^n \alpha^2} \frac{\partial \alpha}{\partial \Psi} \tag{30}$$

and

$$\Delta' = \left\langle E_3'\left\{(3-2m)+\sum_{j=1}^{\infty}A_j+\sum_{j=1}^{\infty}(j-m+1)C_j\right\}+E_3\left\{\sum_{j=1}^{\infty}A_j'+\sum_{j=1}^{\infty}(j-m+1)C_j'\right\}-E_4'\sum_{j=1}^{\infty}(j-m+1)A_j \right.$$
$$\left. -E_4\sum_{j=1}^{\infty}(j-m+1)A_j'-\left(E_2'E_3+E_2E_3'-E_1'E_4-E_1E_4'\right)\Delta\right\rangle \Big/ [E_2E_3-E_1E_4] \tag{31}$$

Here, prime means, partial differentiation with respect to δ and ψ separately and for brevity, expressions of E_1', E_2', E_3' and E_4' are given in Appendix A.

4. Results and discussion

Using the above analysis, the thermal performance of straight taper fins has been investigated for a wide variation of geometrical parameters and Biot number. Some typical results have been reported in this section. The performance of a fin with trapezoidal cross section depends on the angle of taper or the ratio of tip to base thickness (λ). To avoid repetitions all the results have been presented for $\lambda = 0.1$. However, it has been seen that the trend of the results does not change for other values of λ. As the exact nature of the variation of heat transfer coefficient along the fin length is yet unknown, both increasing and decreasing trend of heat transfer coefficient from fin base to fin tip have been considered. The present analysis is valid for both insulated and convective conditions of the fin tip.

The variation of efficiency of different straight taper fins with the fin parameter Z_0 is depicted in Fig. 2. In general, the fin efficiency decreases with the increasing value of Z_0. In case of radial fins, the efficiency decreases with the increasing value of radius ratio at a particular Z_0. Identical trends have been observed [1] for circular fins with other profile shapes.

A comparison between a spine and a longitudinal fin with identical fin parameter, tip to base thickness ratio (λ) as well as the ratio of tip to base heat transfer coefficient (ε) reveals that the spine will have a slightly higher efficiency. However, the difference reduces with the increase of ε. Comparing Fig. 2a and b, it can be seen that an increase in the value of ε increases the fin efficiency.

As one of the purpose of the present work to investigate the effect of non-uniform heat transfer coefficient on the performance of straight taper fins a wide variation ε has been considered. The results are depicted graphically in Fig. 3. The variation ε does not have the uniform effect over the entire range of variation though the efficiency of all the fins increases with the increase of ε. For the convenience of discussion the entire range of ε may be divided into three approximate ranges—lower ($\varepsilon \leqslant 0.2$), intermediate ($0.2 \leqslant \varepsilon \leqslant 4$) and higher ($\varepsilon \geqslant 4$). The rate of increase of efficiency with

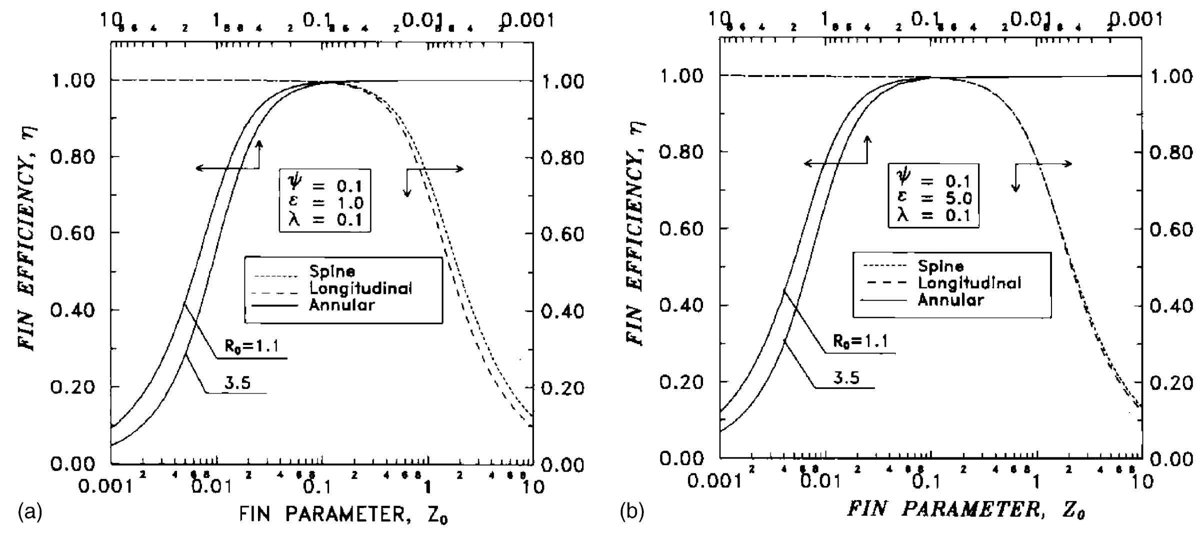

Fig. 2. Relationship between fin efficiency and fin parameter for: (a) $\varepsilon = 1$, (b) $\varepsilon = 5$.

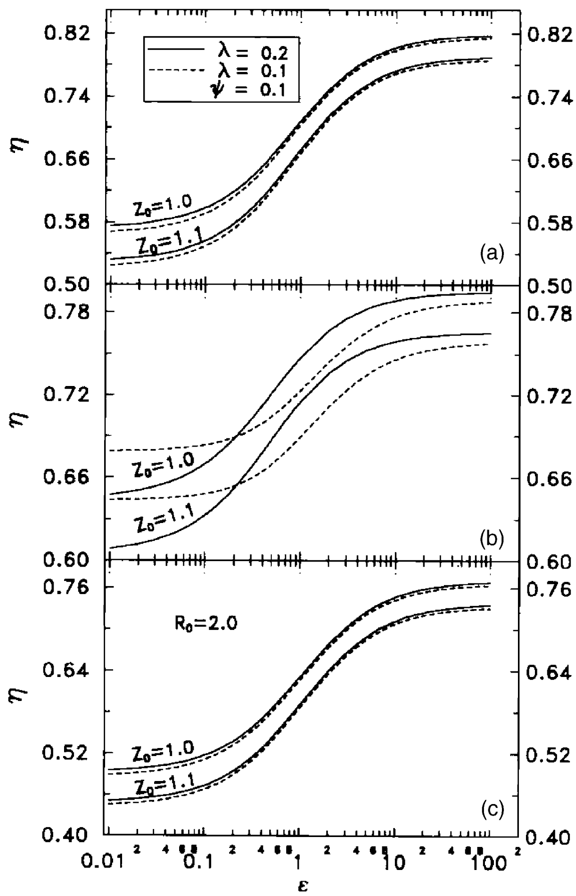

Fig. 3. Effect of non-uniform heat transfer coefficient on fin efficiency: (a) longitudinal fin, (b) spine (c) annular fin.

ε is low in the lower range. The fin efficiency increases at a steeper rate in the intermediate range. The rate of increase of efficiency again falls in the higher range of ε. As the intermediate range of ε is relevant for convective situations observed in practice, it may be concluded that the non-uniform heat transfer coefficient has a profound effect on fin efficiency.

The fin efficiency also depends on the ratio of tip to base thickness λ. It can be seen from curves of Fig. 3a and c that both for longitudinal and annular geometries fin efficiency decreases with λ over the entire range of ε. However, in the case of spines a unique trend is observed. In the lower range of ε, fin efficiency increases with a decrease of λ. In the intermediate and high range of ε, an opposite effect of λ on fin efficiency is noted.

Optimum dimensions of a fin may be obtained either maximizing the rate of heat transfer for a fixed fin volume or minimizing the fin volume for a given rate of heat transmission. This implies Eq. (26) is to be solved simultaneously either with Eq. (20) or with Eq. (25) according to the specification of the problem. In the present investigation fin volume has been taken as the constraint condition.

Results for the optimization studies for the three types of fins under volume constraint are shown in Fig. 4. In these figures, the families of curves depict the variation of the rate of heat dissipation with Bi/ψ, for specified fin volumes. Bi/ψ indicates the Biot number with respect to the fin length. For any type of fins, the rate of heat dissipation initially increases, then reaches a maximum and finally declines with further increase of Bi/ψ. The peaks of the curves indicate the optimum design of the fins for a specified fin volume. It can further be seen that for a given fin volume the optimum design of a fin is strongly influenced by ε. The maximum rate of heat dissipation increases substantially for the same fin volume when ε is increased from 0 to 1.

Often it is required to design the fin for a specified heat duty. Moreover, the designer may be interested to determine the optimum fin dimensions under such a condition. This can easily be done with the analysis presented in section. Based on this analysis, the variation of optimum parameters for longitudinal fins is presented in Fig. 5 for different

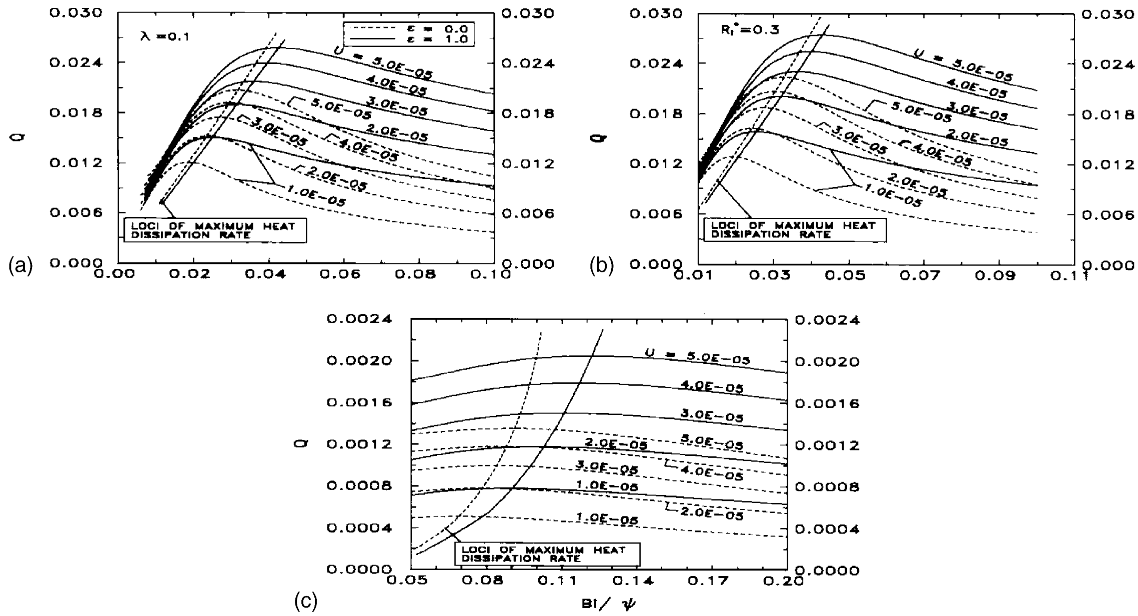

Fig. 4. Heat dissipation rate from straight taper fins as a function of Bi/ψ: (a) longitudinal fin, (b) annular fin, (c) spine.

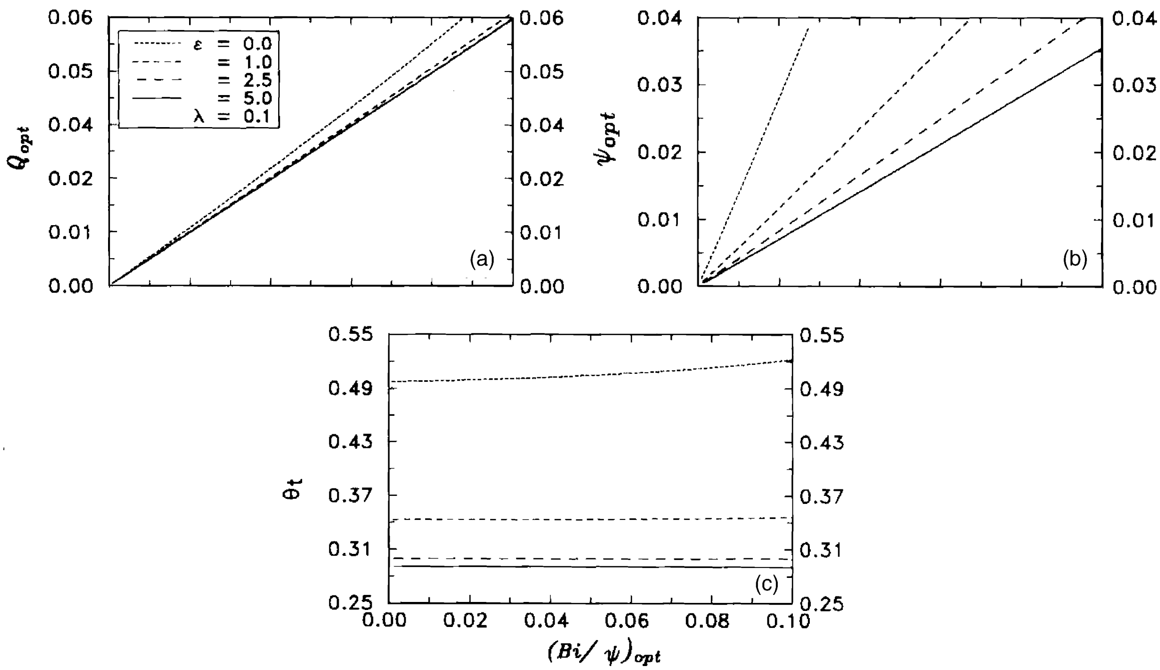

Fig. 5. Parametric values from optimum longitudinal straight taper fins: (a) heat dissipation rate, (b) aspect ratio, (c) tip temperature.

values of ε. For the specified rate of heat dissipation, the optimum value of Bi/ψ for a longitudinal fin can be obtained from Fig. 5a for different values of ε. From Fig. 5b, the required value of ψ for the optimum fin can be determined based on the value of $(Bi/\psi)_{opt}$ obtained from the previous step. If the material property of fin and the average heat transfer coefficient along the fin surface is known, all the important geometrical parameters namely, the fin length, the root

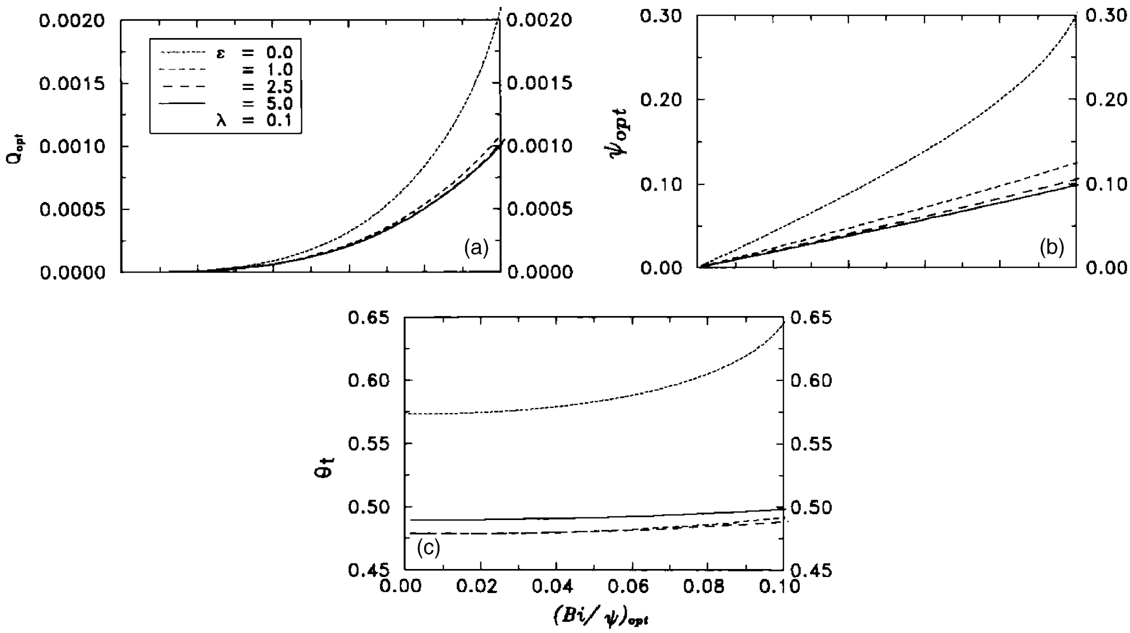

Fig. 6. Parametric values from optimum straight taper spine: (a) heat dissipation rate, (b) aspect ratio, (c) tip temperature.

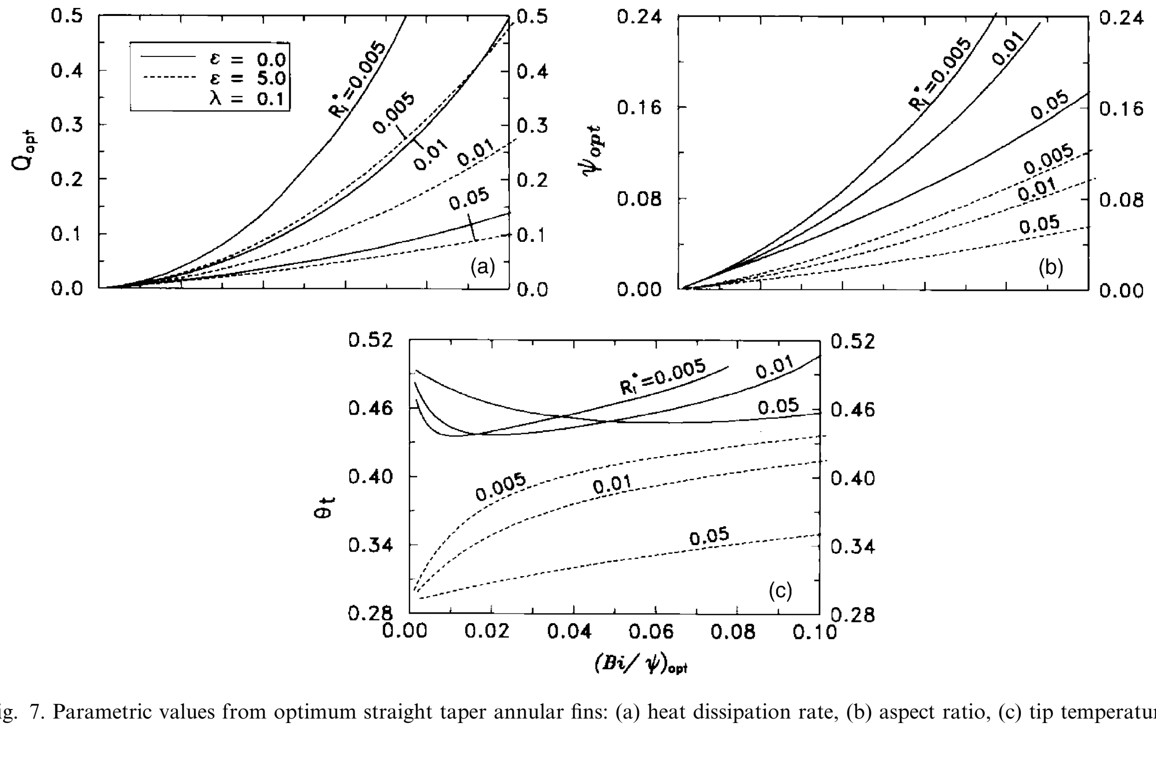

Fig. 7. Parametric values from optimum straight taper annular fins: (a) heat dissipation rate, (b) aspect ratio, (c) tip temperature.

thickness and the tip thickness can be determined along with the fin volume. Fig. 5c gives an additional information regarding the tip temperature (θ_t) of optimum fins. This is not a strict design requirement for most of the applications. However, the information may be useful from the safety considerations of the working personnel [37]. In a similar fashion, Figs. 6 and 7 depict the optimum parameters for spine and annular fins respectively.

5. Conclusions

A methodology for the thermal analysis and optimization of straight taper fins has been discussed in the present paper. The technique adopted for the thermal analysis is a generalized one as it presents a common formulation, set of boundary conditions and method of solution for the three main types of fins, namely longitudinal, spine and annular. The analysis does not make simplifying assumptions like LAI and insulated fin tip. Moreover, it considers a linear variation of heat transfer coefficient along the fin length. Using the Frobenius expanding series, the generalized fin equation has been solved analytically and a single expression has been provided for the temperature distribution in the fins. Based on the thermal analysis the optimum design of the straight taper fins has been obtained from the classical derivative technique. The derived condition of optimality gives an open choice to the designer. For a specified fin volume, optimum fins can be designed to have the maximum rate of heat transfer; alternatively, fins of minimum volume can be designed for a specified heat duty.

It has been seen that the ratio of base to tip heat transfer coefficient (ε) has a profound effect on the fin performance. In general, the efficiency of all the three types of fins increases with the increase of ε. However, the rate of increase is higher in the intermediate range of ε. For a given fin volume the rate of heat dissipation increases also from an optimally designed fin for higher values of ε.

The merit of the technique lies in the fact that its application is not limited to only linear variation of fin thickness and heat transfer coefficient. The method is applicable for fins with arbitrary variation of fin thickness and heat transfer coefficient along the fin length.

Appendix A

$$E'_1 = \sum_{j=1}^{\infty} A'_j \tag{A.1}$$

$$E'_2 = \sum_{j=1}^{\infty} C'_j \tag{A.2}$$

$$E'_3 = \sum_{j=1}^{\infty} (j-m+1)\lambda^{j-m} A'_j - \frac{(2Z_0 Z'_0 \beta + Z_0^2 \beta')}{(1-\lambda)} \left[(2-m) + \sum_{j=1}^{\infty} A_j \lambda^{j-m+1} \right] - \frac{Z_0^2 \beta}{(1-\lambda)} \sum_{j=1}^{\infty} A'_j \lambda^{j-m+1} \tag{A.3}$$

$$E'_4 = \sum_{j=1}^{\infty} A'_j \lambda^{j-m} + \ln\lambda \sum_{j=1}^{\infty} (j-m+1) A'_j \lambda^{j-m} + \sum_{j=1}^{\infty} (j-m+1) C'_j \lambda^{j-m} - \frac{(2Z_0 Z'_0 \beta + Z_0^2 \beta')}{(1-\lambda)} \times \left[\ln\lambda \left\{ 2-m+\sum_{j=1}^{\infty} A_j \lambda^{j-m+1} \right\} + \sum_{j=1}^{\infty} C_j \lambda^{j-m+1} + (m-1)\lambda^{1-m} \right] - \frac{Z_0^2 \beta}{(1-\lambda)} \left[\ln\lambda \sum_{j=1}^{\infty} A_j \lambda^{j-m+1} + \sum_{j=1}^{\infty} C'_j \lambda^{j-m+1} \right] \tag{A.4}$$

$$A'_0 = 0 \tag{A.5}$$

$$A'_1 = m m'_{\rm p} \xi \tag{A.6}$$

$$A'_2 = \langle (2n\alpha' + m m'_{\rm p}\xi) A_1 + (2n\alpha + m m_{\rm p}\xi) A'_1 - m'_{\rm p}(n\alpha\xi + mv) A_0 - n m_{\rm p} \alpha' \xi A_0 - m_{\rm p}(n\alpha\xi + mv) A'_0 \rangle / [2(3-m)] \tag{A.7}$$

$$A'_j = \langle [nj(j-1)\alpha' + m m'_{\rm p}\xi] A_{j-1} + [nj(j-1)\alpha + m m_{\rm p}\xi] A'_{j-1} - m'_{\rm p} \lfloor (n\alpha\xi + mv) A_{j-2} - n\alpha v A_{j-3} \rfloor - m_{\rm p} \lfloor n\alpha' \xi A_{j-2} + (n\alpha\xi + mv) A'_{j-2} \rfloor - nv(\alpha' A_{j-3} + \alpha A'_{j-3}) \rangle / [j(j-m+1)] \tag{A.8}$$

$$C'_0 = 0 \tag{A.9}$$

$$C'_1 = n\alpha' - 2m'_{\rm p}\xi \tag{A.10}$$

$$C_2' = \langle 2(3-m)[3n(\alpha' A_1 + \alpha A_1') + (2n\alpha' + mm_p'\xi)C_1 + (2n\alpha + mm_p\xi)C_1' - (n\alpha\xi + v)\{m_p' C_0 + m_p C_0'\} - m_p n\alpha'\xi C_0\rfloor - (5-m)\lfloor(2n\alpha' + mm_p'\xi)A_1 + (2n\alpha + mm_p\xi)A_1' - (n\alpha\xi + mv)\{m_p' A_0 + m_p A_0'\} - nm_p\alpha'\xi A_0]\rangle / \lfloor 4(3-m)^2\rfloor \quad \text{(A.11)}$$

$$C_j' = \langle j(j-m+1)\{n(2j-1)[\alpha' A_{j-1} + \alpha A_{j-1}'] + [nj(j-1)\alpha' + mm_p'\xi]C_{j-1} + \lfloor nj(j-1)\alpha + mm_p\xi\rfloor C_{j-1}' - (n\alpha\xi + mv)\lfloor m_p' C_{j-2} + m_p C_{j-2}'\rfloor - nm_p\alpha'\xi C_{j-2} + nv(m_p'\alpha C_{j-3} + m_p\alpha' C_{j-3} + m_p\alpha C_{j-3}')\} - (2j-m+1) \times \{\lfloor nj(j-1)\alpha' + mm_p'\xi\rfloor A_{j-1} + \lfloor nj(j-1)\alpha + mm_p\xi\rfloor A_{j-1}' - (n\alpha\xi + mv)\lfloor m_p' A_{j-2} + m_p A_{j-2}'\rfloor - nm_p\alpha'\xi A_{j-2} + nv(\alpha' m_p A_{j-3} + \alpha m_p' A_{j-3} + \alpha m_p A_{j-3}')\}\rangle / [j(j-m+1)]^2 \quad \text{(A.12)}$$

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